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A general approach to linear and non-linear dispersive waves using a Lagrangian

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The basic property of equations describing dispersive waves is the existence of solutions representing uniform wave trains. In this paper a general theory is given for non-uniform wave trains whose amplitude, wave-number, etc., vary slowly in space and time, the length and time scales of the variation in amplitude, wave-number, etc., being large compared to the wavelength and period. Dispersive equations may be derived from a variational principle with appropriate Lagrangian, and the whole theory is developed in terms of the Lagrangian. Boussinesq's equations for long water waves are used as a typical example in presenting the theory.

1. Introduction

In non-linear problems of dispersive waves, solutions taking the form of an infinitely long, periodic wave train are well known. The so-called Stokes waves (Stokes 1847) and cnoidal waves (Korteweg & de Vries 1895) are early examples in the theory of water waves. Recently many similar examples have been found in plasma waves. For problems in one space variable x and the time t, these solutions are found by substituting functions of $x - Ut$ alone for the dependent variables in the governing partial differential equations and solving the resulting ordinary differential equations. Very little beyond these special solutions seems to be known.

In a paper attempting more general solutions (Whitham 1965), an averaging technique was introduced to determine the slow variation in time and space of a non-linear wave train. These slowly varying wave trains occur in two main problems. First of all, it is known from linear theory that an arbitrary initial disturbance 'disperses' into a slowly varying wave train. (In the linear case, the wave train is sinusoidal of course). Secondly, it is often important to consider wave trains entering a slowly varying medium; examples are water waves over a sloping beach or plasma waves propagating through a slowly varying magnetic field. The earlier paper considered the first problem. The notable results were a non-linear generalization of the concept of group velocity and the appearance of a new kind of shock wave. Various mathematical devices were introduced which are similar to those used in classical Lagrangian–Hamiltonian mechanics and it was suggested that more points of similarity would be found. A Lagrangian approach to dispersive waves and to the averaging method has now been obtained and is presented here. At the same time the theory is extended to deal with more space dimensions and to include the propagation in a non-uniform medium.

Although the arguments are general, it seems easiest to give them in the discussion of a specific case and then note the generality of the results afterwards. The Boussinesq equations for long water waves are chosen as a typical and important example. The general theory is discussed in § 9, and the form of the general theory applied to linear systems is given in § 10.

2. Boussinesq equations in water waves

The Boussinesq approximation for long water waves (Boussinesq 1877) is obtained from the exact equations by expanding the velocity potential Φ in a series

$$\Phi(\mathbf{x}, y, t) = \Phi_0(\mathbf{x}, t) + y\Phi_1(\mathbf{x}, t) + y^2\Phi_2(\mathbf{x}, t) + \ldots,$$

where y is the vertical co-ordinate and $\mathbf{x} = (x_1, x_2)$ is the horizontal co-ordinate. It is valid when the ratio of amplitude to depth and the square of the ratio of depth to wavelength are moderately small and of comparable magnitude. For uniform initial depth h_0, the equations are

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x_i}(hu_i) = 0, \tag{1}$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + g\frac{\partial h}{\partial x_i} + \nu \frac{\partial^3 h}{\partial t^2\, \partial x_i} = 0, \tag{2}$$

where $h(\mathbf{x}, t)$ is the depth, $u_i(\mathbf{x}, t)$ is the mean particle velocity, $\nu = \frac{1}{3}h_0$ and g is the acceleration of gravity. The mean velocity u_i is irrotational, as may be seen by taking the curl of (2) or relating it back to Φ; hence, a velocity potential $\phi(\mathbf{x}, t)$ for the mean flow can be introduced such that

$$u_i = \partial\phi/\partial x_i. \tag{3}$$

Equations (1) and (2) then reduce to

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x_i}\left(h\frac{\partial \phi}{\partial x_i}\right) = 0, \tag{4}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x_i}\right)^2 + gh + \nu\frac{\partial^2 h}{\partial t^2} = 0. \tag{5}$$

These equations may be derived from a variational principle

$$\delta \iint L(h, h_t;\ \phi_t, \phi_{x_i})\, d\mathbf{x}\, dt = 0,$$

where the Lagrangian is

$$L = \tfrac{1}{2}\nu h_t^2 - \tfrac{1}{2}gh^2 - h(\phi_t + \tfrac{1}{2}\phi_{x_i}^2). \tag{6}$$

The Euler equations,

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \phi_t}\right) + \frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial \phi_{x_i}}\right) = 0, \tag{7}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial h_t}\right) - \frac{\partial L}{\partial h} = 0, \tag{8}$$

give (4) and (5), respectively.

Conservation laws of the form

$$\partial P/\partial t + \partial Q_i/\partial x_i = 0,$$

play an important role. In terms of the Lagrangian they are

$$\frac{\partial}{\partial t}\left(\phi_t \frac{\partial L}{\partial \phi_t} + h_t \frac{\partial L}{\partial h_t} - L\right) + \frac{\partial}{\partial x_i}\left(\phi_t \frac{\partial L}{\partial \phi_{x_i}} + h_t \frac{\partial L}{\partial h_{x_i}}\right) = 0, \tag{9}$$

$$\frac{\partial}{\partial t}\left(\phi_{x_j} \frac{\partial L}{\partial \phi_t} + h_{x_j} \frac{\partial L}{\partial h_t}\right) + \frac{\partial}{\partial x_i}\left(\phi_{x_j} \frac{\partial L}{\partial \phi_{x_i}} + h_{x_j} \frac{\partial L}{\partial h_{x_i}} - L\,\delta_{ij}\right) = 0, \tag{10}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \phi_t}\right) + \frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial \phi_{x_i}}\right) = 0, \tag{11}$$

$$\frac{\partial}{\partial t}(\phi_{x_j}) - \frac{\partial}{\partial x_j}(\phi_t) = 0, \quad \frac{\partial}{\partial x_i}(\phi_{x_j}) - \frac{\partial}{\partial x_j}(\phi_{x_i}) = 0. \tag{12}$$

Equation (9) is the energy equation and follows from the invariance of L with respect to an arbitrary increase in t, equation (10) is the jth component of momentum and follows from the invariance of L with respect to an arbitrary increase in x_j, equation (11) is the conservation of mass and may be deduced here from the absence of ϕ itself in L, i.e. the invariance of L to increases in ϕ. These are all applications of Noether's theorem (Noether 1918; Courant & Hilbert 1953).

3. The uniform-wave-train solution

In a uniform wave train, $h(\mathbf{x},t)$ and $u_i(\mathbf{x},t)$ are functions of a single phase variable

$$\theta = \kappa_i x_i - \omega t, \tag{13}$$

where $\boldsymbol{\kappa}$ and ω are the wave-number and frequency. Since ϕ itself is not a physical quantity, the most general wave train is obtained by taking

$$\phi = \Phi(\theta) + \beta_i x_i - \gamma t, \quad h = H(\theta), \tag{14}$$

where β_i, γ are constants. There is a certain lack of uniqueness in the definition of β_i, γ, κ_i, ω; any multiple of θ would do equally well and a term linear in θ can always be absorbed into Φ. To fix the definition, the conditions

$$[\theta] = 1, \quad [\Phi] = 0, \tag{15}$$

are imposed, where square brackets denote the increase in the variable over one complete period. (It is anticipated that the solution will be periodic.) Thus, Φ is periodic and the phase θ increases at a rate one per cycle.†

When this form of solution is substituted in (4) and (5) the equations can be integrated to

$$\kappa^2 H \Phi_\theta - (\omega - \beta_i \kappa_i) H = b, \tag{16}$$

$$\begin{aligned} \nu\omega^2 H_\theta^2 &= \kappa^{-2} H^{-1}\{b + (\omega - \beta_i \kappa_i) H\}^2 + (2\gamma - \beta^2) H - gH^2 + 2a, \\ &\equiv F(H), \text{ say}, \end{aligned} \tag{17}$$

† The idea of introducing both ω and κ_i, rather than simply taking $\theta = \alpha_i x_i - t$ and later calculating the period as $-[\theta]$, is crucial in simplifying the later calculations. It was suggested by Mr Jon C. Luke of the California Institute of Technology.

where a and b are constants of integration. It is seen that the solution is periodic with H oscillating between a pair of zeros H_1, H_2 of the function $F(H)$, and

$$\theta = \nu^{\frac{1}{2}}\omega \int_{H_1}^{H} \frac{dH}{\sqrt{F(H)}}. \tag{18}$$

The conditions (15) giving relations between the constants κ_i, ω, β_i, γ, a, b are

$$[\theta] = 2\nu^{\frac{1}{2}}\omega \int_{H_1}^{H_2} \frac{dH}{\sqrt{F(H)}} = 1, \tag{19}$$

and, from (16),

$$[\Phi] = \oint \Phi_\theta d\theta = 2\nu^{\frac{1}{2}}\omega \int_{H_1}^{H_2} \frac{b+(\omega-\beta_i\kappa_i)H}{\kappa^2 H\sqrt{F(H)}}\, dH = 0. \tag{20}$$

One can think of prescribing the wave-number κ_i, the mean velocity β_i, the mean height and the amplitude (which are essentially measured by b and a). Then (19) and (20) determine the other two parameters.

4. The averaged equations for a slowly varying wave train

As explained in §1, there are interesting problems requiring the study of a slowly varying wave train. In the theory developed here, it is assumed that the solution is given locally by the uniform solution (16), (17), (18) but that the parameters κ_i, ω, β_i, γ, a, b are now slowly varying functions of $\mathbf{x}$ and t. Then partial differential equations are obtained for these functions by an appropriate averaging of the original equations. The motivation of the whole approach is discussed in great detail in the earlier paper (Whitham 1965), and the averaged equations are obtained in various examples. However, the procedure can be very much simplified and given deeper significance.

First, it is noted that in the non-uniform wave train the phase function $\theta(\mathbf{x}, t)$ will be related to $\kappa_i(\mathbf{x}, t)$, $\omega(\mathbf{x}, t)$ by

$$\omega = -\partial\theta/\partial t, \quad \kappa_i = \partial\theta/\partial x_i, \tag{21}$$

rather than (13); (21) reduces correctly to (13) when ω and κ_i are constant. Similarly, a phase-like function ψ must be introduced as the generalization of $\beta_i x_i - \gamma t$ in (14) so that

$$\gamma = -\partial\psi/\partial t, \quad \beta_i = \partial\psi/\partial x_i. \tag{22}$$

Next the average Lagrangian

$$\mathscr{L}(\omega, \kappa_i, a;\, \gamma, \beta_i, b) = \int_0^1 L\,d\theta \tag{23}$$

is calculated for the steady profile solution. It turns out to be

$$\mathscr{L} = 2\nu^{\frac{1}{2}}\omega \int_{H_1}^{H_2} \sqrt{F(H)}\,dH - a. \tag{24}$$

Now consider the 'averaged variational principle'

$$\delta \iint \mathscr{L}(\omega, \kappa_i, a; \gamma, \beta_i, b)\, d\mathbf{x}\,dt = 0. \tag{25}$$

Remembering the definitions (21) and (22), the Euler equations for this variational problem are as follows.

$$\theta \text{ variation:} \quad \frac{\partial}{\partial t}\mathscr{L}_\omega - \frac{\partial}{\partial x_i}\mathscr{L}_{\kappa_i} = 0, \tag{26}$$

$$\psi \text{ variation:} \quad \frac{\partial}{\partial t}\mathscr{L}_\gamma - \frac{\partial}{\partial x_i}\mathscr{L}_{\beta_i} = 0, \tag{27}$$

$$a \text{ variation:} \quad \mathscr{L}_a = 0, \tag{28}$$

$$b \text{ variation:} \quad \mathscr{L}_b = 0. \tag{29}$$

The functional relation in (28) is easily verified from (24) to be (19), and relation (29) is (20). Equations (21), (22), (26), (27) provide partial differential equations for the determination of the slowly varying functions.

There are a number of interesting ways of writing the equations. One can eliminate θ and ψ in (21) and (22) to take

$$\left.\begin{aligned} \partial\kappa_i/\partial t + \partial\omega/\partial x_i = 0, \quad \partial\kappa_i/\partial x_j - \partial\kappa_j/\partial x_i = 0,\\ \partial\beta_i/\partial t + \partial\gamma/\partial x_i = 0, \quad \partial\beta_i/\partial x_j - \partial\beta_j/\partial x_i = 0,\end{aligned}\right\} \tag{30}$$

with (26) and (27). Alternatively, one can introduce a Legendre transformation

$$\Omega = \mathscr{L}_\omega, \quad K_i = \mathscr{L}_{\kappa_i}, \quad \Gamma = \mathscr{L}_\gamma, \quad B_i = \mathscr{L}_{\beta_i},$$
$$\mathscr{H}(\Omega, K_i, \Gamma, B_i) = \omega\mathscr{L}_\omega + \kappa_i\mathscr{L}_{\kappa_i} + \gamma\mathscr{L}_\gamma + \beta_i\mathscr{L}_{\beta_i} - \mathscr{L};$$

then $\omega = \partial\mathscr{H}/\partial\Omega$, etc., and a canonical form of (21), (22), (26), (27) becomes

$$\partial\theta/\partial t = -\partial\mathscr{H}/\partial\Omega, \quad \partial\theta/\partial x_i = \partial\mathscr{H}/\partial K_i, \quad \partial\Omega/\partial t - \partial K_i/\partial x_i = 0, \tag{31}$$

$$\partial\psi/\partial t = -\partial\mathscr{H}/\partial\Gamma, \quad \partial\psi/\partial x_i = \partial\mathscr{H}/\partial B_i, \quad \partial\Gamma/\partial t - \partial B_i/\partial x_i = 0. \tag{32}$$

This is a Hamiltonian form with momenta Ω, K_i for the co-ordinate θ and momenta Γ, B_i for the co-ordinate ψ. The right-hand sides of the last equations in each set would be $\partial\mathscr{H}/\partial\theta$ and $\partial\mathscr{H}/\partial\psi$, respectively, in the general case, but this is the special case when the co-ordinates are cyclic and $\mathscr{H}$ does not depend explicitly on θ and ψ.

5. Adiabatic invariants

The slowly varying wave train in continuum mechanics is analogous to the problems treated by the theory of adiabatic invariants in the mechanics of finite systems. In that theory the problems concern the change in amplitude, etc., of an oscillating system as some external parameter is varied slowly with time. The theory of adiabatic invariants can also be obtained from an average Lagrangian, although it is not usually done in that way (see, for example, Landau & Lifshitz 1960 for a discussion).

Consider a system with one degree of freedom and suppose the Lagrangian is $L(q, \dot{q}, \lambda)$, where $\lambda(t)$ is a slowly varying function of the time. When λ is constant, it is assumed that the solution is periodic. In that case, there is also an energy integral

$$\dot{q}\frac{\partial L}{\partial \dot{q}} - L \equiv \dot{q}p - L = E, \tag{33}$$

where the total energy E is a constant of integration. This relation can be solved to find p as a function of q, λ and E. The average Lagrangian is

$$\begin{aligned}\mathscr{L}(\omega, E, \lambda) &= \frac{1}{T}\int_0^T L\,dt, \\ &= \frac{1}{T}\int_0^T p\dot{q}\,dt - E, \\ &= \omega \oint p\,dq - E, \end{aligned} \tag{34}$$

where T is the period, $\omega = T^{-1}$ is the frequency, and $\oint$ denotes the integral over one cycle. Since $p = p(q, \lambda, E)$,

$$\oint p\,dq = I(\lambda, E).$$

Remembering that the frequency ω is the derivative $\dot{\theta}$ of a phase function $\theta(t)$, the variational principle

$$\delta \int \mathscr{L}\,dt = 0$$

gives

$$\frac{d}{dt}\mathscr{L}_\omega = 0, \quad \mathscr{L}_E = 0, \tag{35}$$

in perfect correspondence with (26) to (29). The first equation in (35) gives the adiabatic invariant $\mathscr{L}_\omega$ and the second equation gives a functional relation for the frequency. From (34), they become

$$\mathscr{L}_\omega = \oint p\,dq = I(\lambda, E) = \text{const.}, \tag{36}$$

$$T = 1/\omega = \partial I/\partial E. \tag{37}$$

These are the well-known results in mechanics.

Now the equation (26) (and similarly (27)) can be interpreted as the balance between the changes of a time-like adiabatic invariant $\mathscr{L}_\omega$ with time and the changes of a space-like adiabatic invariant $\mathscr{L}_{\kappa_i}$ in space.

For general finite mechanical systems with more than one degree of freedom the classical theory of adiabatic invariants relies heavily on the Hamiltonian formulation and canonical transformations to cyclic variables. The form of the results given in (31), (32) seems closely related with that approach. However, a *derivation* of (31) and (32) following that approach seems to be ruled out by a theorem of Rüssman (1961) showing that the only canonical transformations for Hamiltonian systems with more than one independent variable are point transformations.

6. Extension to propagation in a non-uniform medium

For a wave-train incident on a slowly varying medium, the equations obtained from the variational principle still hold. Even though $\mathscr{L}$ depends on position $\mathbf{x}$ explicitly (in addition to the dependence implicitly through the functions ω, κ_i, a, etc.), the Euler equations (26) to (29) are unchanged. Thus, for example, the changes in amplitude due to a non-uniform depth $h_0(\mathbf{x})$ can be studied for the non-linear-water-wave problem. The parameters ω, κ_i, a, γ, β_i, b will be functions of $\mathbf{x}$ alone, and the time derivatives drop out of the averaged equations. In particular, for propagation in one space dimension the solution to the problem is given by

$$\left.\begin{aligned} \mathscr{L}_\kappa = \text{const.}, \quad \mathscr{L}_\beta &= \text{const.}, \\ \mathscr{L}_a = \mathscr{L}_b &= 0, \\ \omega = \text{const.}, \quad \gamma &= \text{const.}, \end{aligned}\right\} \tag{38}$$

the constants being determined from the incident wave. Notice that the space-like adiabatic invariants $\mathscr{L}_\kappa, \mathscr{L}_\beta$ are constant, in close analogy with the time-dependent problems of mechanics.

The detailed evaluation of the results and their physical or practical significance in the theory of water waves is postponed until a later paper. This paper is devoted to the general formulation rather than to detailed applications.

7. Conservation laws

In the paper referred to earlier (Whitham 1965), the averaged equations governing the wave train were obtained by averaging the conservation equations† (9) to (12). The function W used extensively in that paper is essentially the integral occurring in (24); it was introduced as a pseudo-adiabatic invariant and its connexion with $\mathscr{L}$ is analogous to (34). Even with the introduction of $\mathscr{L}$, the direct calculation of the average fluxes and densities is relatively long. However, it is straightforward, and the averaged forms of equations (9) to (12) are

$$\frac{\partial}{\partial t}(\omega\mathscr{L}_\omega + \gamma\mathscr{L}_\gamma - \mathscr{L}) - \frac{\partial}{\partial x_i}(\omega\mathscr{L}_{\kappa_i} + \gamma\mathscr{L}_{\beta_i}) = 0, \tag{39}$$

$$-\frac{\partial}{\partial t}(\kappa_j\mathscr{L}_\omega + \beta_j\mathscr{L}_\gamma) + \frac{\partial}{\partial x_i}(\kappa_j\mathscr{L}_{\kappa_i} + \beta_j\mathscr{L}_{\beta_i} - \mathscr{L}\delta_{ij}) = 0, \tag{40}$$

$$-\frac{\partial}{\partial t}\mathscr{L}_\gamma + \frac{\partial}{\partial x_i}\mathscr{L}_{\beta_i} = 0, \tag{41}$$

$$\partial\beta_j/\partial t + \partial\gamma/\partial x_j = 0, \quad \partial\beta_j/\partial x_i - \partial\beta_i/\partial x_j = 0, \tag{42}$$

respectively. Of course, these are readily and most satisfactorily obtained from (25). Equations (39), (40), (41) may be seen from the invariance of $\mathscr{L}$ with respect to increases in t, x_j, ψ, respectively.

Equation (27) is identical with (41) and, when the derivatives in (39) to (42) are expanded, equation (26) is obtained after trivial elimination. However, the equations relating κ_i and ω, as in (30), are not found immediately from this conservation-law approach. In the first instance, it is shown from (39) to (42) that

$$\mathscr{L}_\omega(\partial\kappa_j/\partial t + \partial\omega/\partial x_j) - \mathscr{L}_{\kappa_i}(\partial\kappa_j/\partial x_i - \partial\kappa_i/\partial x_j) = 0. \tag{43}$$

Then, together with the identity

$$\frac{\partial}{\partial t}(\partial\kappa_j/\partial x_i - \partial\kappa_i/\partial x_j) = \frac{\partial}{\partial x_i}(\partial\kappa_j/\partial t + \partial\omega/\partial x_j) - \frac{\partial}{\partial x_j}(\partial\kappa_i/\partial t + \partial\omega/\partial x_i), \tag{44}$$

† These hold for a uniform medium and have additional terms on the right for a non-uniform medium.

it is deduced that κ_i remains irrotational if it is so initially; thus

$$\partial\kappa_j/\partial t + \partial\omega/\partial x_j = 0, \quad \partial\kappa_j/\partial x_i - \partial\kappa_i/\partial x_j = 0. \tag{45}$$

When shock solutions are considered, i.e. discontinuities in the variables ω, κ_i, a, γ, β_i, b are allowed, the corresponding shock conditions must be taken from the true conservation equations (39) to (42), not from the adiabatic equation (26) or from the conservation of waves (45). This was explained and discussed in detail in §6 of the earlier paper.

8. Group velocity and energy propagation

A velocity may be derived from any conservation equation by dividing the flux by the density. Probably the most important one comes from the energy equation (39)

$$\text{energy velocity} = -\frac{\omega\mathscr{L}_{\kappa_i} + \gamma\mathscr{L}_{\beta_i}}{\omega\mathscr{L}_\omega + \gamma\mathscr{L}_\gamma - \mathscr{L}}. \tag{46}$$

However, the differential equations (26) to (30) turn out to be hyperbolic, and characteristic velocities, governing the propagation of changes in the quantities $\omega, \kappa_i, a, \gamma, \beta_i, b$, can be calculated. Mathematically these are even more important. In linear theory, the characteristic velocities and the energy velocity coincide in the usual group velocity; this is shown later in §10. In non-linear problems the characteristic velocities certainly provide the correct generalization of group velocity as far as kinematic properties are concerned. This was discussed in more detail in the earlier paper.

9. Theory for general systems

Dispersive systems stem from Lagrangians with certain special properties. In the Boussinesq problem, for example, the Lagrangian (6) does not involve ϕ itself; only the derivatives of ϕ appear. It seems that the general result is: the Lagrangian for a dispersive system may be written in terms of n functions χ, $\phi^{(\alpha)}$ $(\alpha = 1, \ldots, n-1)$ and their first derivatives, but the functions $\phi^{(\alpha)}$ are potentials and only their derivatives occur. The uniform-wave-train solution will have the main phase function

$$\theta = \kappa_i x_i - \omega t,$$

and $n-1$ subsidiary phase-like functions

$$\psi^{(\alpha)} = \beta_i^{(\alpha)} x_i - \gamma^{(\alpha)} t \quad (\alpha = 1, \ldots, n-1)$$

and it will take the form

$$\chi = \chi(\theta),$$

$$\phi^{(\alpha)} = \Phi^{(\alpha)}(\theta) + \psi^{(\alpha)}.$$

The form is specified uniquely by demanding

$$[\theta] = 1, \quad [\Phi^{(\alpha)}] = 0.$$

The average Lagrangian $\mathscr{L}$ will be a function of κ_i, ω, $\beta_i^{(\alpha)}$, $\gamma^{(\alpha)}$ and n integration constants a, $b^{(\alpha)}$. The averaged equations for a non-uniform wave train are obtained from

$$\delta \iint \mathscr{L}\, d\mathbf{x}\, dt = 0,$$

with
$$\left.\begin{aligned} \omega = -\theta_t, \qquad \kappa_i = \theta_{x_i}, \\ \gamma^{(\alpha)} = -\psi_t^{(\alpha)}, \quad \beta_i^{(\alpha)} = \psi_{x_i}^{(\alpha)}. \end{aligned}\right\} \tag{47}$$

The Euler equations for the variational problem are then

$$\left.\begin{aligned} \frac{\partial}{\partial t}(\partial \mathscr{L}/\partial \omega) - \frac{\partial}{\partial x_i}(\partial \mathscr{L}/\partial \kappa_i) = 0, \\ \frac{\partial}{\partial t}(\partial \mathscr{L}/\partial \gamma^{(\alpha)}) - \frac{\partial}{\partial x_i}(\partial \mathscr{L}/\partial \beta_i^{(\alpha)}) = 0, \\ \partial \mathscr{L}/\partial a = 0, \quad \partial \mathscr{L}/\partial b^{(\alpha)} = 0. \end{aligned}\right\} \tag{48}$$

The case of a slowly varying non-uniform medium is automatically included when $\mathscr{L}$ depends explicitly on $\mathbf{x}$ and t, but the equations (47), (48) are unchanged.

The restriction of the form of the Lagrangian presumably corresponds to the assumption of separability of the Hamilton–Jacobi equation in the classical theory of adiabatic invariants.

10. General linear systems

For linear systems, the periodic wave train is sinusoidal and the functions χ, $\Phi^{(\alpha)}$ are sinusoidal. The constants a, $b^{(\alpha)}$ can be taken as their amplitudes. The $\beta_i^{(\alpha)}$ and $\gamma^{(\alpha)}$ are trivial constants added to the physical quantities

$$\partial \phi^{(\alpha)}/\partial x_i, \quad \partial \phi^{(\alpha)}/\partial t$$

and there is no loss in omitting them. (This is certainly not true for non-linear systems.) The amplitudes $b^{(\alpha)}$ are easily determined in terms of a from substitution in the governing equations. If the $\beta_i^{(\alpha)}$, $\gamma^{(\alpha)}$, $b^{(\alpha)}$ are eliminated in this way, the average Lagrangian, being quadratic in the amplitudes, must be reducible to the form

$$\mathscr{L}(\omega, \kappa_i, E) = G(\omega, \kappa_i)\, E, \quad E = \tfrac{1}{2}a^2. \tag{49}$$

The Euler equation for the variation in E is

$$\mathscr{L}_E = G(\omega, \kappa_i) = 0, \tag{50}$$

and this is just the usual dispersion relation between ω and κ_i for linear dispersive waves. It should be noted that this functional relation can also be written $\mathscr{L} = 0$.

With
$$\omega = -\theta_t, \quad \kappa_i = \theta_{x_i},$$

the other Euler equation for the variation in θ gives

$$\frac{\partial}{\partial t}\mathscr{L}_\omega - \frac{\partial}{\partial x_i}\mathscr{L}_{\kappa_i} = 0.$$

After substitution for $\mathscr{L}$ and elimination of θ, the equations for E, ω, κ_i reduce to

$$\left.\begin{aligned} \frac{\partial}{\partial t}(EG_\omega) - \frac{\partial}{\partial x_i}(EG_{\kappa_i}) = 0, \\ \partial \kappa_i/\partial t + \partial \omega/\partial x_i = 0, \quad \partial \kappa_i/\partial x_j - \partial \kappa_j/\partial x_i = 0, \\ G(\omega, \kappa_i) = 0. \end{aligned}\right\} \tag{51}$$

The group velocity C_i is defined by

$$C_i(\boldsymbol{\kappa}) = d\omega/d\kappa_i = -G_{\kappa_i}/G_\omega, \tag{52}$$

and the differential equations become

$$\left.\begin{aligned}\frac{\partial \mathscr{E}}{\partial t}+\frac{\partial}{\partial x_i}(\mathscr{E}C_i) &= 0,\\ \frac{\partial \kappa_i}{\partial t}+C_j\frac{\partial \kappa_i}{\partial x_j} &= 0,\end{aligned}\right\} \tag{53}$$

for $\mathscr{E} = EG_\omega$ and κ_i. The two characteristics of equations (53) coincide, and the characteristic velocity is equal to the group velocity C_i.

The energy equation is

$$\frac{\partial}{\partial t}(\omega\mathscr{L}_\omega-\mathscr{L})-\frac{\partial}{\partial x_i}(\omega\mathscr{L}_{\kappa_i}) = 0.$$

Therefore, the energy velocity is

$$-\omega\mathscr{L}_{\kappa_i}/(\omega\mathscr{L}_\omega-\mathscr{L}) = -G_{\kappa_i}/G_\omega = C_i,$$

and the energy velocity coincides with the characteristic velocity, both being equal to the group velocity.

11. Linear example

One of the simplest linear examples which occurs in a number of physical problems is

$$\frac{\partial^2\phi}{\partial t^2}-c^2\frac{\partial^2\phi}{\partial x_i^2} = \mu^2\frac{\partial^4\phi}{\partial x_i^2\partial t^2}. \tag{54}$$

The linearized Boussinesq equations reduce to this with $c^2 = gh_0$, $\mu^2 = \frac{1}{3}h_0^2$. It is equivalent to the pair of second-order equations

$$\left.\begin{aligned}\phi_t-c^2\chi-\mu\chi_{tt} &= 0,\\ \chi_t-\phi_{x_ix_i} &= 0,\end{aligned}\right\} \tag{55}$$

which may, in turn, be obtained from the Lagrangian

$$L = \tfrac{1}{2}\mu^2\chi_t^2+\chi\phi_t-\tfrac{1}{2}\phi_{x_i}^2-\tfrac{1}{2}c^2\chi^2. \tag{56}$$

The uniform periodic wave train is

$$\left.\begin{aligned}\chi &= a\sin\theta, \quad \theta = \kappa_i x_i-\omega t,\\ \phi &= b\cos\theta,\end{aligned}\right\} \tag{57}$$

where the possible additional term $\beta_i x_i-\gamma t$ in ϕ has been omitted. On substitution in (55), the amplitudes are found to be related by

$$b = \omega\kappa^{-2}a, \tag{58}$$

and the dispersion relation is

$$\omega^2 = \frac{c^2\kappa^2}{1+\mu^2\kappa^2}, \quad \kappa^2 = \kappa_i^2. \tag{59}$$

From (56) and (57), the average Lagrangian is found to be

$$\mathscr{L} = \tfrac{1}{4}(\mu^2\omega^2 - c^2)\,a^2 + \tfrac{1}{2}\omega ab - \tfrac{1}{4}\kappa^2 b^2. \tag{60}$$

The Euler equations for a and b are

$$\mathscr{L}_a = \tfrac{1}{2}(\mu^2\omega^2 - c^2)\,a + \tfrac{1}{2}\omega b = 0,$$
$$\mathscr{L}_b = \tfrac{1}{2}\omega a - \tfrac{1}{2}\kappa^2 b = 0,$$

and these lead to (58) and (59). Alternatively, using (58) to eliminate b in (60), we have

$$\mathscr{L} = \frac{1}{2}\,\frac{\omega^2(1+\mu^2\kappa^2) - c^2\kappa^2}{\kappa^2}\,E, \quad E = \tfrac{1}{2}a^2,$$

and the dispersion relation is $\mathscr{L} = 0$, all in accord with the general form in §10.

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